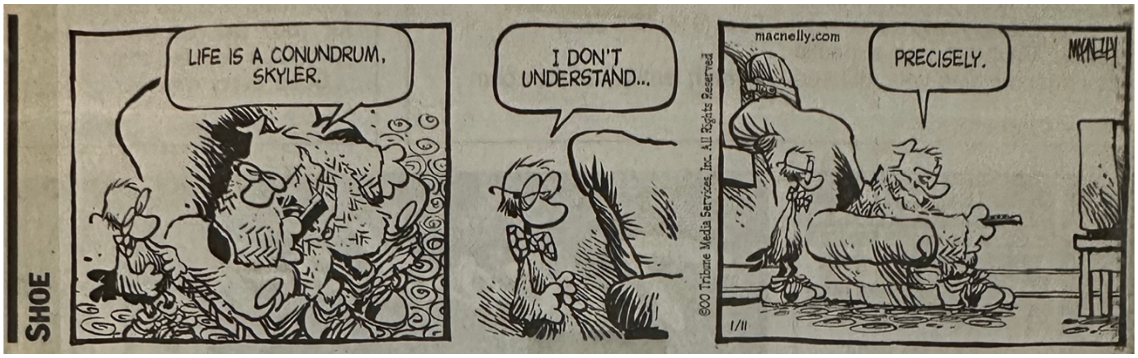
Chapter 18

"Indistinguishable"



Credit – Shoe: "Conundrum" (need permission)

Statistics, the dismal math. It's hard enough, what with our intuition often being at odds, with, well you know, odds, but now the quantum wants in the game, with one objective, to change the rules. At the classical level, everything is distinguishable from everything else. Two objects may look alike but look closer and there are always differences. A nick, a smudge, a stain, just one atom out of place. Every snowflake is different.

Not so in the quantum domain. In this bizarre place, two particles can be, in a formal mathematical sense, indistinguishable, and violate our intuition in a whole new way. Indistinguishable particles change the statistics of measurements. I swear, you can't make this stuff up.

Indistinguishability

Consider two fair coins, to be flipped once, each, independently. Easy-peasy. Each coin has an even chance of heads or tails, neither outcome depends on the other coin, so there are just four possibilities, HH, HT, TH, TT. Each equally likely, 25%. The odds of them matching is, trivially, 50%. Too obvious to even mention – if they are classical coins.

But if they are quantum coins, a complication sets in. They might be indistinguishable. In the standard model, particles can be divided into two categories: fermions and bosons. Fermions have half integer spin; bosons have integral spins. This has strange consequences; two fermions cannot be in the same state at the same time. This is fortuitous, for electrons are fermions, and this Pauli exclusion rule keeps them from all running downhill into the lowest orbital around an atom's nucleus. All of chemistry depends on this restriction.

Bosons, on the other hand, have no problem being in the same state, and photons are bosons. For a pair of indistinguishable particles, one cannot tell one from another, so the middle two options collapse into just one. You can tell *how many* boson-like coins came up heads, but you can't tell *which one* it is. Therefore, when they are tossed, there are not four possibilities, there are only three. The statistics are different.

Notation

For a quantum system, one with eigenstates and eigenvectors, we will need an improved notation. Let's build what we need in stages. For quantum coins that are distinguishable the system state for a pair prior to measurement would be represented in a four-dimensional Hilbert space like this.

$$\Psi_{12} = \frac{1}{\sqrt{4}}\begin{Bmatrix} \pm|H\rangle_1|H\rangle_2 \pm |H\rangle_1|T\rangle_2 \\ \pm|T\rangle_1|H\rangle_2 \pm |T\rangle_1|T\rangle_2 \end{Bmatrix} \tag{1}$$

For quantum coins that are indistinguishable the anti-diagonal gets condensed, where we can't pretend to know which ket goes with which particle. The system state for a pair prior to measurement would be represented in a three-dimensional Hilbert space. We need a way to clearly specify the joint eigenvectors. Here is a first attempt.

$$\Psi_{12} = \frac{1}{\sqrt{3}}\{\pm|H\rangle_1|H\rangle_2 \pm |h\rangle_{12}|t\rangle_{21} \pm |T\rangle_1|T\rangle_2\} \tag{2}$$

The use of lowercase labels and multiple indices on the antidiagonal kets (term 2) is intended to represent that the indistinguishable states have condensed to one eigenvector, but it's confusing, and not very scalable.

The goal of a scalable notation offers some representational clues. Consider an upgrade to three classical coins, that's an 8-dimensional Hilbert space. For three indistinguishable coins, that condenses to a 4-dimensional Hilbert space: all heads, 2 heads, 1 head, and no heads.

$$\Psi_{123} = \frac{1}{\sqrt{4}}\{\pm|H\rangle_1|H\rangle_2|H\rangle_3 \pm |h\rangle|t\rangle|h\rangle \pm |t\rangle|h\rangle|t\rangle \pm |T\rangle_1|T\rangle_2|T\rangle_3\} \tag{3}$$

But if we don't need subscripts on the inner terms, we don't need them on the outer terms either.

$$\Psi_{123} = \frac{1}{\sqrt{4}}\{\pm|H\rangle|H\rangle|H\rangle \pm |h\rangle|t\rangle|h\rangle \pm |t\rangle|h\rangle|t\rangle \pm |T\rangle|T\rangle|T\rangle\} \tag{4}$$

The mix of uppercase and lowercase is a little weird, but easily fixed, it's just that which way to go is arbitrary. The real problem is that while the subscripts have been dropped, there will still be a tendency to equate sequential position with a particular particle which could be easily confused with an entangled state.

Returning to the case of just two indistinguishable particles, let's try this. It is more concise, and equally scalable.

$$\Psi_{12} = \frac{1}{\sqrt{3}}\{\pm|H2\rangle \pm |H1\rangle \pm |H0\rangle\} \tag{5}$$

Or for you Australians

$$\Psi_{12} = \frac{1}{\sqrt{3}}\{\pm|T0\rangle \pm |T1\rangle \pm |T2\rangle\} \tag{6}$$

The major challenge of this notation is that both versions fail to communicate what the alternative pure eigenstate is. We need to combine equations (5) & (6). As a final standard, try this.

$$\Psi_{12} = \frac{1}{\sqrt{3}}\{\pm|H2\rangle \pm |H1T1\rangle \pm |T2\rangle\} \tag{7}$$

To confirm scalability, again consider the three-particle case.

$$\Psi_{123} = \frac{1}{\sqrt{4}}\{\pm|H3\rangle \pm |H2T1\rangle \pm |H1T2\rangle \pm |T3\rangle\} \tag{8}$$

This is readable, concise, and scalable. Add T0 and H0 if you feel compelled to do so.