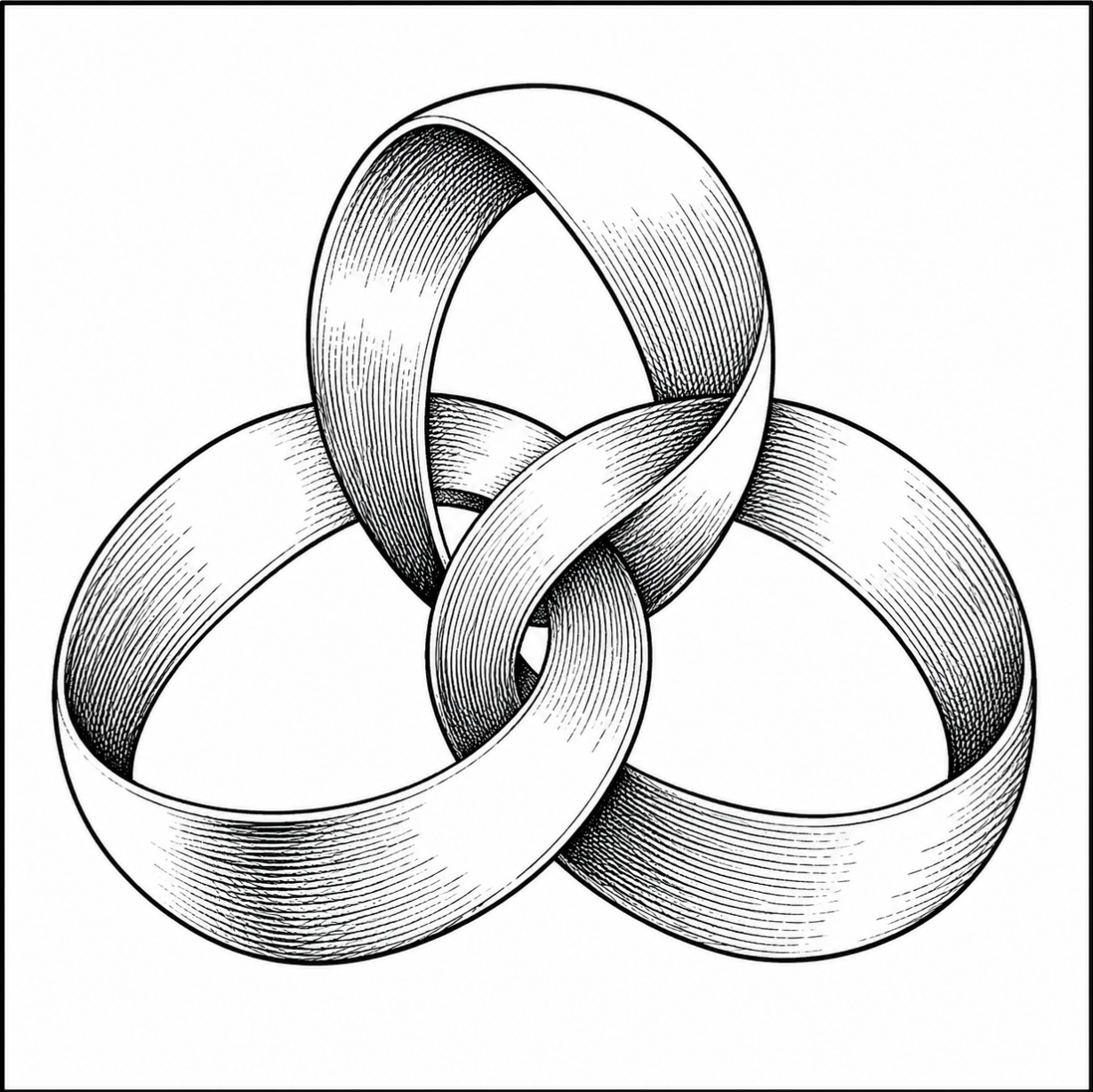


## Chapter 25

### “LOF – Laws of Form”



*Mobius Borromean*

“A lot of work...”

A lesser author might have panicked – I merely procrastinated. For my part of this dialog, I have sought clear pedagogy, to make complex concepts as intuitive and meaningful as possible. But that doesn’t lessen the territory to cover, just makes the grade a little less steep. The following chapters are audacious. We are going to attempt to derive logic from first principles. And since it

seems that we'll have to use logic to do that, at least if we want to have confidence that our reasoning is valid, then we are in a catch 22 – we need logic in order to derive it. One more example of self-reference, nay even meta self-reference.

But if you are like Paradigm, you might find the epiphanies worth the trek. You might also consider the wisdom in Curiosity's quip, "He's not afraid of a little hard work, he can lay right down beside it and go to sleep."

When the new concepts feel overwhelming, give your mind a chance to renormalize, to reintegrate the old knowledge with the new. The subconscious part of our minds is good at that. Take a break. If necessary, take a nap.

## Laws of Form

Now the good news; this has already been done before. Long before in fact – 1969. In a small concise book titled, "Laws of Form" the eccentric British mathematician G. Spencer-Brown succeeded in *deriving* Boolean logic from first principles. This stands in sharp contrast to George Boole who in the 19<sup>th</sup> century *postulated* the principles of logic that now bear his name. Boole invented a couple of operators and a couple of operands and coupled them with a handful of axioms, and thus created the first symbolic logic. Our modern high-tech industries, digital circuits, computers, software, the entire digital age – all trace their roots back to good ole' Boole.

And G. Spencer-Brown did him one better.

Here's how he did it. He starts at such a masterful level of abstraction that the very concepts of operand and operator are degenerate. That's like saying '1' and '+' are the same thing. It is pure unadulterated genius, and it is my privilege to not only walk you through it, but also (immodestly) to extend his work.

## The Mark

To begin; "You cannot make a distinction without making an indication." G. Spencer-Brown then introduces the *mark*, this symbol

$$\lrcorner \quad (1)$$

as the indication of a distinction.

## Axioms

Ok... so what? We've indicated something, but what? Should we treat the *mark* as a noun or as a verb? In the first case, a noun is just a label like a name. If someone calls you twice, such as, "Sam, come here, now Sam, now" the calling has not changed your name, Sam is still your label. Thus,

"To re-call is to call." Or, symbolically

$$\lrcorner \lrcorner = \lrcorner \quad (A1)$$

This is our first rule of logic, our first axiom. If we have a form where the identical thing (think noun) is duplicated, we can eliminate the duplication. Call this the axiom of *Condensation*.

Now for the other interpretation of the mark, as a verb. Picture a circle drawn in the plane. It divides the plane into two regions, one inside the circle, the other outside the circle. The circle (indication) distinguishes the inside from the outside. A verb is an action, such as an instruction to cross the circle. Did we cross into the circle, or out from the circle? Doesn't yet matter, we don't know.

But we do know that "to re-cross is not to cross." Symbolically

$$\overline{\neg} = \quad (A2)$$

We now have our second rule of logic, our second axiom. If we have a form where the identical thing (think verb) is nested, we can eliminate both marks. Call this the axiom of *Cancellation*.

What's interesting here is that we have yet to even create a symbol for true, nor one for false either, for the matter, but we do clearly have only two possible states; marked or unmarked. Pick either one to represent true – it's arbitrary. Or, to tease poor Mathematics, the interpretation is multi-valued. Same as for the square root of minus one.

## Arithmetic

With these two rules we can now compute logical values. This is not yet a proof system, our axioms do not yet lead to theorems, that will require an *algebra* of logic; but what we do have at this stage is an *arithmetic* of logic.

As an example, consider this expression.

$$\overline{\overline{\neg}} \quad (2)$$

Apply the axiom of Condensation.

$$\neg \neg = \neg \quad (3)$$

Then apply the axiom of Cancellation.

$$\neg = \quad (4)$$

Voila, we end up with the unmarked state.



$$\neg \neg = . \quad (5)$$

Calculation is easy, and since every step simplifies the form, is guaranteed to terminate.

## Operand/Operator Degeneracy

Now the claim at the start of this chapter was that operand and operator could be degenerate. As an operator, the mark is clearly a negate operator. For a single input, it is just the NOT operator, while for two inputs it is the NOR operator. It should be clear that the ‘space’ within which the marks exist is a kind of *or* space, with an unlimited number of inputs. As an operand, one is free to choose the *mark* as either true or false; pick a standard. Thus, the *mark* is both operand and operator. How brilliant.

For those inquisitive readers familiar with Boolean logic, the NOR operator can be used to construct expressions which represent all the other logical operators that Boole postulated. Thus, G. Spencer-Brown *derived* logic from first principles. He showed that given a *distinction*, logic follows.

## An Algebra

In order to show how self-reference shows up in logic, we have to move beyond an arithmetic to an algebra. G. Spencer-Brown does this meticulously with full mathematical rigor over several chapters, but the path is tedious, and in general rather obvious; obvious in the sense that there are no surprises. All the canons and theorems line up with common sense. He also establishes consistency, completeness and independence; important concepts, but ones which pretty much only a mathematician cares about. For our purposes we can leave the proof of these properties to the professionals.

A few salient points to make clear and then we will be ready to derive imaginary truthvalues.

First, not having a symbol for the unmarked state is awkward. In decimal math, the invention of the zero, and a symbol for it ‘0’, greatly aided both calculation and understanding. Thus, we shall also introduce two symbols; ‘m’ for the marked state, and ‘n’ for the not marked state. (I know, I know, *unmarked* is better English, but ‘m’ and ‘n’ are next to each other alphabetically, and they are the symbols chosen by G. Spencer-Brown.) We chose to stick with his standard here. Note that this step breaks the operand/operator degeneracy. The values ‘m’ and ‘n’ are now our operands, and the *mark* is the operator. Note its generic nature, it can be a unary operator, a binary operator, even an n-input operator; just depends on the number of symbols it ‘covers’.

Second, we introduce letters which are intended to represent arbitrary finite expressions. Using them, theorems can be derived that allow for faster calculation. If we are confined to only the two arithmetical axioms, one can compute, but only rather slowly. It’s like using the tick mark system (|||); suitable for the basic operation of counting, but tedious if you want to do multiplication.

Third, expressions which have equivalent values, will be coupled with an equal sign. Thus,

$$\overline{\overline{p|p}} = n \quad (J1)$$

Fourth, we take the excursion to infinity. To do this requires the application of 4 theorems. Each is almost trivial in its simplicity, and while we'll skip the proofs (the perfectionist reader is directed to *Laws of Form*) it is easy to use the two arithmetical axioms to demonstrate that each theorem is valid.

Here they are (using G. Spencer-Brown's overly complicated taxonomical scheme), and using his names for them:

Transposition

$$\overline{\overline{pr|qr}} = \overline{\overline{p|q}}r \quad (J2)$$

Reflexion

$$\overline{\overline{a}} = a \quad (C1)$$

Occultation

$$\overline{\overline{a|b}}a = a \quad (C4)$$

Iteration

$$aa = a \quad (C5)$$

Then what we are going to do is apply them in five steps to this form,

$$\overline{\overline{a|b}} = ? \quad (6)$$

(But of course, not in the above order, that would be too simple; sigh.) The order will be C5, C1, J2, C4 and C1. Thus...

$$= \overline{\overline{a|b}}\overline{\overline{a|b}} \quad (C5)$$

$$= \overline{\overline{a|b}}\overline{\overline{\overline{\overline{a|b}}}} \quad (C1)$$

Now let a be p, NOT b be q, and r be NOT (NOT a OR b) the original expression (6). Then we can use (J2)

$$= \overline{\overline{\overline{a|b|a|a|b|b}}}$$
 (J2)

$$= \overline{\overline{\overline{a|b|a|b}}}$$
 (C4)

$$= \overline{\overline{\overline{a|b|a|b}}}$$
 (C1)

And then once again, just to show that this process can be repeated ad infinitum...)

$$= \overline{\overline{\overline{\overline{\overline{a|b|a|b|a|b}}}}}$$
 (C5)

$$= \overline{\overline{\overline{\overline{\overline{a|b|a|b|a|b}}}}}$$
 (C1)

$$= \overline{\overline{\overline{\overline{\overline{\overline{\overline{a|b|a|a|b|b|a|b}}}}}}}$$
 (J2)

$$= \overline{\overline{\overline{\overline{\overline{\overline{\overline{a|b|a|b|a|b}}}}}}}$$
 (C4)

$$= \overline{\overline{\overline{\overline{\overline{\overline{\overline{a|b|a|b|a|b}}}}}}}$$
 (C1)

It should be clear that this process can be continued indefinitely generating an increasingly long sequence of alternating a's & b's.

$$= \dots a | b | a | b \quad (7)$$

Let f symbolize this unending sequence.

$$f = \dots a | b | a | b \quad (8)$$

Then we can recast as a reentrant form.

$$f = f a | b \quad (9)$$

And there is the self-reference.

### Indeterminacy

If you chose not to follow the derivation in detail, that's ok. The important equations are (8) (9) where an infinite expression is more concisely expressed as a self-referential one. Because the expression is infinite there is no reason to presume that it will have the same value as any of the finite length precursors, perhaps it does, but perhaps it does not. Since all the axioms assumed finite forms, they cannot be employed to answer this question. What can be done however, is to perform an exhaustive analysis of the possible values of a and b, which since each has only two potential values (marked or unmarked) means we need to try just four permutations. Immanently feasible.

Here they are...

$$f m | m = n \quad (10)$$

$$f m | n = m \quad (11)$$

$$f n | m = n \quad (12)$$

$$f n | n = m \text{ or } n \quad (13)$$

